

Monster symmetry and Extremal CFTs

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Abstract

We test some recent conjectures about extremal selfdual CFTs, which are the candidate holographic duals of pure gravity in AdS_3 . We prove that no $c = 48$ extremal selfdual CFT or SCFT may possess Monster symmetry. Furthermore, we disprove a recent argument against the existence of extremal selfdual CFTs of large central charge.

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1 Introduction

The author of [6] argues that the holographic dual of pure (super)gravity in three dimensions should be a holomorphically factorized $c = 24k$ ($c = 12k^*$) (super)conformal field theory with as few low dimension (super)Virasoro primaries as possible. Modular invariance alone prescribes a unique factorized torus partition functions with no Virasoro primary field of dimension smaller or equal to $\frac{c}{24}$. It is not known if an “Extremal” CFT with such a partition function actually exists for $k > 1$, nor if it is unique: some non-trivial checks on the existence of low- k ECFTs are available [5–7], but [4] formulates a conjecture which implies the non-existence of $k > 42$ ECFTs. The only known examples, the $k = 1$ ECFT and the $k^* = 1, 2$ ESCFT, possess a very large discrete symmetry group: the most notable is the $k = 1$ theory, which possesses two copies of the Monster discrete symmetry group, each acting on one chiral half of the theory. It is conjectured in [6] that the Monster group might be a symmetry of 3d quantum gravity in general. Also, the tentative partition function for a $k^* = 4$ theory also suggests Monster symmetry [6].

The purpose of this note is to test the above conjectures. In the first section we prove that no $k = 2$ ECFT may possess Monster symmetry. Our strategy is simple: If a ECFT exists which has Monster symmetry, it must be possible to define twist fields σ_g and twisted partition functions $Z_g = \text{Tr } g q^{L_0}$ for each symmetry group element g in the Monster. It turns out to be impossible to consistently build such objects for group elements in the $2A$ conjugacy class of the Monster group. The approach works as well for the $k^* = 4$ theory, which is discussed in the second section. Our strategy does not allow a clear cut

conclusion about higher k ECFTs. The partial results we present here should still be useful in constraining the symmetry structure of generic ECFTs. Additionally, in the last section we will present a counterexample to the conjecture in [4].

The motivation for this work is quite straightforward. The only strategy available at this time to construct ECFTs with $k > 1$ is direct conformal bootstrap. It is conceivable that an ECFT may be defined as a W algebra by a finite set of OPEs for a choice of Virasoro primary fields. There are some indications, for example, that a conformal bootstrap is possible for the $k = 1$ Monster module. The dimension 2, 3, 4, 5 primaries all sit in single irreducible representations of Monster symmetry, and some normal ordered products of the dimension 2 fields and their derivatives do sit in the same irreps as the dimension 4 and 5 fields. If those normal ordered products are not null vectors, the singular part of the OPE of dimension 2 and 3 fields will involve only other dimension 2 and 3 fields and their normal ordered products. This defines a W -algebra. Because of Monster symmetry, the only unknown coefficients in the OPE are the overall normalization of the three point functions. Associativity and unitarity of the W -algebra can be reduced to a set of Fierz identities for the appropriate Monster representations. One may hope for a similar structure of OPEs in a higher k ECFTs, which close on the fields of level from $k + 1$ to $2k + 1$. The large number of fields involved even for very small k makes it clear that only a large discrete symmetry group will make the conformal bootstrap possible in practice. This makes it important to find some way to test for the existence of Monster symmetry in the next simplest example, $k = 2$, and possibly find a prescription of which Monster irreps would contain the level 3 to 5 fields, to jump-start a bootstrap procedure. Unfortunately our negative results make this strategy unfeasible, unless a different candidate is found for a large symmetry group. On the other hand the counterexample presented in the last section of this note makes it clear that the existence of extremal CFTs is still a possibility for all k .

2 $k = 2$ ECFT

A ECFT with a chiral symmetry group G has chiral g -twisted partition functions $Z_g(\tau) = \text{Tr } g q^{L_0}$ for each $g \in G$, which have interesting modular properties. For example, the set of twisted partition functions $t_g(\tau)$ for the the Monster module is quite famous, and is identical to the set of genus zero hauptmodules. This fact was instrumental in the discovery of the Monster module itself. The modular group transforms the partition functions $Z_g(\tau)$ are a set of more general Z_{g^a, g^b} , defined as a torus partition function with g^a and g^b inserted at the two cycles of the torus. The action of the modular group is quite intuitive:

$$S : (a, b) \rightarrow (b, -a) \qquad T : (a, b) \rightarrow (a, a + b) \qquad (1)$$

If an anomaly is present for the chiral group symmetry, the modular group acts projectively, and extra phases appear in the transformation rules. The coefficients of the twisted partition

functions $Z_g = Z_{e,g}$ must decompose into characters of the symmetry group appropriately: if $Z(\tau) = \sum_R d_R f_R(\tau)$ then $Z_{e,g} = \sum_R \chi_R(g) f_R(\tau)$. On the other hand the coefficients of the partition functions of twisted sectors Z_{g^a, g^b} must decompose into characters of the stabilizer of g^a in the symmetry group.

The first constraint on the existence of a ECFT with Monster symmetry is that such twisted partition functions must exist for all the Monster group conjugacy classes, be compatible with each other and satisfy certain positivity and integrality requirements. (The untwisted partition function of a twisted sector $Z_{g,e}$ must have positive integers as coefficients, up to a global ambiguous phase). This is not quite enough to fix the partition functions uniquely. As described in the appendix A, there is a standard recipe to build candidates as “twisted” Hecke transforms of the hauptmodules $t_g(\tau)$. On the other hand the analysis of modularity will be enough to determine the value of the dimension of the ground state in the $2A$ twisted sector of the $k = 2$ ECFT. This value will turn out to be inconsistent with the OPE of the twisted sector ground state with itself.

The modular properties of a Z_2 twisted partition function are described by the following diagram:

$$Z_g \Leftrightarrow_T Z_g \Leftrightarrow_S Z^g \Leftrightarrow_T Z_g^g \Leftrightarrow_S Z_g^g \quad (2)$$

The modular transformations above may in general be anomalous, so that phases are generated through the transformations. An example of this phenomenon is evident in the list of Hauptmodules t_g . For example t_{3C} is $Z_{3C} = J(3\tau)^{\frac{1}{3}}$, the S transform of it is $Z^{3C} = J(\tau/3)^{\frac{1}{3}}$, which has a power expansion involving powers $q^{\frac{n}{3} - \frac{1}{9}}$. Under T^3 this partition function of the twisted sector goes back to itself as expected, but with a phase of $\frac{2\pi}{3}$.

The anomalous phases are constrained by the requirement that $S^2 = 1$ and $(ST)^3 = 1$. In certain cases, in particular for the case at hand of a Z_2 symmetry, these two equations fix all phases up to some discrete choices. Let us fix a few irrelevant phases by defining Z^g as the S transform of Z_g and Z_g^g as the T transform of Z^g . The S transform of Z_g^g should be Z_g^g itself up to a phase, and $S^2 = 1$ fixes the phase to be $\epsilon = \pm 1$. The T transform of Z_g^g should be a multiple of Z^g again, up to a phase α . $(ST)^3 = 1$ acting over Z_g gives $\epsilon\alpha = 1$. Hence there are two possibilities: if $\epsilon = 1$, Z^g will have a q expansion involving integer and half integer powers of q ; if $\epsilon = -1$ Z^g will have a q expansion involving powers of the form $q^{n \pm 1/4}$. After the precise modular transformation rules are known, the set of partition functions can be organized into a vector-valued holomorphic modular form. Such forms are uniquely specified (up to a possible constant term) by the set of all their polar terms, which are the coefficients of the negative powers of q . For a unitary theory of central charge 48 and $\epsilon = 1$ the polar terms are

$$Z_g = \frac{1}{q^2} + \frac{a_1}{q} + a_0 + \mathcal{O}(q) \quad Z^g = \frac{b_{3/2}}{q^{3/2}} + \frac{b_1}{q} + \frac{b_{1/2}}{q^{1/2}} + \mathcal{O}(1) \quad (3)$$

The constant term a_0 is a “polar term” iff all the anomalous phases are zero. If $\epsilon = -1$ the

polar terms are

$$Z_g = \frac{1}{q^2} + \frac{a_1}{q} + \mathcal{O}(1) \quad Z^g = \frac{b_{7/4}}{q^{7/4}} + \frac{b_{5/4}}{q^{5/4}} + \frac{b_{3/4}}{q^{3/4}} + \frac{b_{1/4}}{q^{1/4}} + \mathcal{O}(q^{3/4}) \quad (4)$$

A basis of modular forms for the $\epsilon = -1$ case can be readily generated:

$$\frac{E_6(2\tau)}{\eta(2\tau)^{12}} = \frac{1}{q} + \mathcal{O}(1) \quad S \left[\frac{E_6(2\tau)}{\eta(2\tau)^{12}} \right] = -\frac{1}{q^{1/4}} + \mathcal{O}(q^{3/4}) \quad (5)$$

$$\frac{E_6(\tau)}{\eta(2\tau)^{12}} = \frac{1}{q} + \mathcal{O}(1) \quad S \left[\frac{E_6(\tau)}{\eta(2\tau)^{12}} \right] = -\frac{64}{q^{1/4}} + \mathcal{O}(q^{3/4}) \quad (6)$$

$$64 \frac{E_6(\tau)\eta(2\tau)^{12}}{\eta(\tau)^{24}} = \mathcal{O}(1) \quad S \left[64 \frac{E_6(\tau)\eta(2\tau)^{12}}{\eta(\tau)^{24}} \right] = -\frac{1}{q^{3/4}} + \frac{12}{q^{1/4}} + \mathcal{O}(q^{3/4}) \quad (7)$$

Any other modular form with these modular properties can be reproduced from its polar terms as a linear combination of those three with coefficients polynomials of J . Interestingly, for all k we cared to test (up to $k = 13$), and in particular for $k = 2$, if the polar terms of Z_g are set to the coefficients of the vacuum Virasoro module and the polar terms of Z^g are integer, the constant coefficient a_0 turns out to be a multiple of 8. On the other hand for $k < 10$ the number of Virasoro vacuum descendants of dimension k is not a multiple of 8. This is a contradiction, and implies that no consistent anomalous Z_2 twisted partition function can be written for an ECFT of central charge $24k$, $k < 10$.

The non-anomalous case requires extra information from the OPEs. Let us start with an instructive example: the OPE of Z_2 twist fields in the case of the $k = 1$ ECFT. The $2B$ twisted partition function can be written in terms of the discriminant as

$$Z_{2B} = T_{2B} = \frac{\Delta(\tau)}{\Delta(2\tau)} - 24 = \frac{1}{q} + 276q - 2048q^2 + 11202q^3 - 49152q^4 + 184024q^5 + \mathcal{O}(q^6) \quad (8)$$

The modular transform of it is the partition function of a $2B$ twisted sector.

$$Z^{2B} = 2^{12} \frac{\Delta(\tau)}{\Delta(\tau/2)} - 24 = 24 + 4096\sqrt{q} + 98304q + 1228800q^{3/2} + 10747904q^2 + \mathcal{O}(q^{5/2}) \quad (9)$$

Notice that the 24 twisted sector ground states have dimension one, and that their OPE is in the untwisted sector. The singular part must be

$$\sigma_i(z) \sigma_j(0) \sim \frac{\delta_{ij}}{z^2} \quad (10)$$

Hence they are 24 independent free bosons. This is quite obviously consistent with the fact that the $2B$ orbifold of the Monster CFT is the same as the Leech Narain lattice model.

On the other hand the $2A$ twisted partition function is

$$Z_{2A} = T_{2A} = \frac{\Delta(\tau)}{\Delta(2\tau)} + 2^{12} \frac{\Delta(2\tau)}{\Delta(\tau)} - 24 = \frac{1}{q} + 4372q + 96256q^2 + 1240002q^3 + \mathcal{O}(q^4) \quad (11)$$

The modular transform of it is the partition function of a $2A$ twisted sector.

$$Z^{2A} = 2^{12} \frac{\Delta(\tau)}{\Delta(\tau/2)} + \frac{\Delta(\tau/2)}{\Delta(\tau)} - 24 = \frac{1}{\sqrt{q}} + 4372\sqrt{q} + 96256q + 1240002q^{3/2} + \mathcal{O}(q^2) \quad (12)$$

The single $2A$ twist field must have OPE

$$\sigma(z) \sigma(0) \sim \frac{1}{z} \quad (13)$$

and is a free fermion.

The first regular term in the OPE of a free fermion with itself is a $c = \frac{1}{2}$ energy momentum tensor T_σ . It is a well known fact that among the level two primaries of the $k = 1$ theory one can always pick a set of free-fermion energy momentum tensors.

It should be clear that this sort of consideration becomes very constraining at higher k , where no level two primaries exist. For example the Z_2 twist fields cannot have dimension $\frac{1}{2}$ or 1, otherwise they would be free fermions and free bosons, and generate extra energy momentum tensors with central charge $\frac{1}{2}$ or one, while at level 2 only a single energy momentum tensor of $c = 24k$ exists. If the twisted sector ground state is a dimension $3/2$ field, the OPE with itself will contain the total energy momentum tensor, and will give rise to a superconformal algebra. As there are no dimension 1 currents in the untwisted sector, this algebra can have $N = 1$ at most, hence there can only be up to one twisted sector ground state of dimension $3/2$. So we have learned that $b_{3/2} = b_1 = 0$, $b_{1/2} = 0$ or 1.

The partition functions can be written in terms of polynomials of J acting on 1, $\frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}}$ and $\frac{\eta(2\tau)^{24}}{\eta(\tau)^{24}}$, and will be

$$Z_g = 1 + \frac{1}{q^2} + (-4096 + 4096b_{1/2})q + (98580 + 98304b_{1/2})q^2 \quad (14)$$

And

$$\begin{aligned} Z^g &= \frac{b_{1/2}}{\sqrt{q}} + 25 - 24b_{1/2} + (196608 + 276b_{1/2})\sqrt{q} + \\ &+ (21495808 - 2048b_{1/2})q + (864288768 + 11202b_{1/2})q^{3/2} \end{aligned} \quad (15)$$

Positivity of the coefficients of Z^g also confirms $b_{1/2} = 0$ or 1 . Correspondingly, the partition function Z_g will be either $T_{2B}^2 - 551$ or $J(2\tau) + 1$. The first case is a perfect candidate for Z_{2B} . The second case is *not* a good candidate for Z_{2A} , as it gives zero as the character for the dimension 3 primaries, but all the characters for $2A$ in the lower irreps of the monster are positive numbers. Hence the $k = 2$ theory cannot have Monster symmetry

What can we say for higher values of k ? Already at $k = 3$ the constraint is much weaker. Rough candidate partition functions can be readily generated with a single spin field ground state of dimension $3/2$, which forms a Super Virasoro algebra with the energy momentum tensor. A specific candidate really shines through: the one whose polar terms contain one spin field of dimension $3/2$ and one of dimension 2 . The coefficients decompose appropriately in characters of the Monster, the coefficients of the modular transform decompose neatly in dimensions of the centralizer of $2A$. Before concluding that this partition function provides strong evidence for Monster symmetry at $k = 3$ though, the reader is invited to check Appendix A, where a simple physically motivated construction, the twisted Hecke transform, is described. The construction makes it clear that there is a standard way to generate candidate twisted partition functions for a theory with any possible spectrum of polar untwisted states, such that all the partition functions are automatically consistent with each other in the decomposition in characters of the Monster, and such that their modular transforms decompose nicely in dimensions and characters of the centralizer. This fact is a direct consequence of the existence of the $k = 1$ Monster module, so the existence of a nice candidate for a ECFT twisted partition function is hardly a surprise. It is possible that the requirements of positivity of the coefficients of partition functions in twisted sectors will still have some strength, but we will not pursue the matter further in this note. Interestingly, our methods can be used to exclude such deceptively beautiful candidates for $k > 3$. Spin fields of dimension $3/2$ or 2 are not allowed anymore, as their normal ordered product would be a non-existent non-Virasoro dimension 4 untwisted field. Twist fields of dimension up to $k/2$ can also be excluded, as they would generate non-existent W-algebras. [1]

3 $k^* = 4$ ESCFT

A similar reasoning can be applied to the partition function of extremal SCFTs. The modular structure of twisted partition functions is slightly more complicated. There are nine partition functions for a Z_2 symmetry g , with three choices of spin structure and three of g -twist. There are two orbits under the modular group. The smaller orbit has size three, it involves the Ramond partition function with a g insertion $Z_{NS,g}^R$, the partition function in the g -twisted NS sector with $(-1)^F$ inserted, $Z_R^{NS,g}$, and the g -twisted NS partition function with a g insertion $Z_{NS,g}^{NS,g}$,

$$Z_{NS,g}^R \Leftrightarrow_T Z_{NS,g}^R \Leftrightarrow_S Z_R^{NS,g} \Leftrightarrow_T Z_{NS,g}^{NS,g} \Leftrightarrow_S Z_{NS,g}^{NS,g} \quad (16)$$

This triple of partition functions has very similar properties to the ones considered in the previous section, the $Z_{NS,g}^R$ is integer moded, and $Z_R^{NS,g}$, $Z_{NS,g}^{NS,g}$ will have moding $q^n, q^{n+1/2}$ in the non anomalous case, $q^{n\pm 1/4}$ in the anomalous case. The other six partition functions form a separate, cyclic orbit, with the NS partition function with g insertion $Z_{NS,g}^{NS}$ sent by T to $Z_{R,g}^{NS}$, sent by S to $Z_{NS}^{R,g}$, sent by T to $Z_{NS,g}^{R,g}$, sent by S to $Z_{R,g}^{NS,g}$, sent by T to $Z_{NS}^{NS,g}$, sent by S back to the beginning of the orbit. We will not make any use of this orbit in the following section.

Let us collect the information available about the polar terms in the non anomalous case. First of all, in any R sector there should be no polar terms. Following the example of the bosonic theory one can exclude twist fields of dimension $1/2, 1$ in NS g-twisted sectors and cap the number of fields with dimension $3/2$ to one, because the OPE of two NS twist fields gives untwisted fields in the NS sector, which for level smaller than 3 are only the identity and e.m. tensor. Furthermore we know that there is only one Ramond ground state, which has to be g-invariant if the Monster is a symmetry of the theory. The orbit of size three has polar terms only from the NS g-twisted fields, hence if there are no twist fields of dimension $3/2$ the partition function will just be constant, one. Inspection of the characters of the $2A$ and $2B$ classes shows that this cannot be the case for the Ramond partition function with $2A$ or $2B$ insertion. If there is one field of dimension $3/2$, then the resulting partition function actually agrees well with the result expected from a $2B$ twist, with the coefficients of $Z_{NS,g}^R$ decomposing nicely into the expected $2B$ characters. Again, the $2A$ twisted partition function cannot be non-anomalous. To reproduce the expected $2A$ character of dimension 3 Ramond fields one would need 47 dimension $3/2$ NS $2A$ -twisted fields, and then the rest of the partition function would still not decompose properly into $2A$ characters. The anomalous case also does not work, for the same reason as in the previous section. The constant coefficient in $Z_{NS,g}^R$ turns out to be again $8(b_{1/4} + 4b_{3/4} + 6b_{5/4})$ in terms of the polar coefficients of $Z_R^{NS,g}$. Hence it cannot be one, as it should be for the ESCFT. We conclude that the $k^* = 4$ ESCFT cannot have monster symmetry.

4 Size of a chiral algebra vs order of minimal monic differential constraints

The author of [4] is interested in an argument of [8] relating a certain algebraic property of a chiral algebra \mathcal{A} , the size s , to the existence of a monic differential operator of degree s with coefficients in polynomials of the Eisenstein series E_2, E_4, E_6 which annihilates the

characters of all the representations of \mathcal{A} . The size is defined as the smallest degree of a monic polynomial in L_{-2} which sits in a certain subspace \mathcal{O}_q of the vacuum representation \mathcal{H}_0 of \mathcal{A} . If a chiral algebra has size s then L_{-2}^s will always sit in another important subspace called $\mathcal{O}_{[2]}$. $\mathcal{O}_{[2]}$ is spanned by vectors of the form $\{A_{-h_A-1}|v\rangle \mid A \in \mathcal{A}, |v\rangle \in \mathcal{H}_0\}$. The important conjecture of [4] is essentially the converse of the above result: the minimum degree of a monic differential operator which annihilates all the characters of the chiral algebra should be the same as the size of the algebra. The conjecture rules out ECFTs for high enough k , because the partition function of ECFTs of high k is annihilated by monic differential operators of degree smaller than k , but the size of the chiral algebra of an ECFT is bigger than k .

We can build a simple counterexample to this claim: it is the n -th power of the Monster module. The monster module has size 3: L_{-2}^3 sits in $\mathcal{O}_{[2]}$ while L_{-2}^2 does not. Correspondingly the partition function $J(q)$ of the monster CFT is annihilated by a monic differential operator of degree 3. The size of the n -th power of the Monster CFT is at least $2n+1$: $(L_{-2}^{tot})^{2n+1} = (\sum L_{-2}^i)^{2n+1}$ sits in $\mathcal{O}_{[2]}$, as it is the sum of terms each of which contains at least one L_{-2}^i raised to the power of 3 or more, which sits in $\mathcal{O}_{[2]}$; on the other hand $(L_{-2}^{tot})^{2n} \sim \prod_i (L_{-2}^i)^2 + v$, $v \in \mathcal{O}_{[2]}$, which is not in $\mathcal{O}_{[2]}$. (A similar argument for the theory e_8^l is given in [4])

By inspection for the first few values of n (up to $n=15$), it is possible to show that J^n satisfies a monic differential equation of degree $n+2$ and no monic equation of degree $n+1$. Already for $n=2$ this degree is smaller than the size. Essentially, the partition function may satisfy “accidental” differential equations, which are not due to singular vectors in the language of [4]. Hence we see no obstacle to have ECFTs with arbitrarily high k .

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A Twisted Hecke transforms

The most natural way to describe the partition function Z of any selfdual CFT with $c=24k$ is to use Hecke transforms of the J function, which is the partition function of the Monster CFT, [6], as

$$Z = \sum a_i H_i J(\tau) \tag{17}$$

Here the coefficients a_i are the coefficients of the polar terms. This means that the coefficients of the partition function of any selfdual CFT with $c = 24k$ decompose naturally in dimensions of Monster group representations. Clearly the Monster is not the symmetry group of all selfdual CFTs, and one might believe that the existence of a consistent set of Monster-twisted partition functions would be a much stronger requirement, able to discriminate candidates with Monster symmetry. In this Appendix we mean to show that a concept of "twisted Hecke transforms" exists, which automatically builds a set of Z_g which are consistent with Z and with each other.

For a physicist, the Hecke transforms of the partition function F of a CFT, $H_n F(\tau)$, are defined as the building blocks for the partition function of symmetric product orbifolds of that CFT. The chiral partition functions for a chiral symmetric product of a selfdual CFT can be computed by the methods of [3] and collected in a generating function

$$U(p, q) = \sum_N p^N Z^{\text{Sym}N}(\tau) = \frac{1}{\prod_{n>0} (1 - p^n q^n)^{c(nm)}} \quad (18)$$

Here $F(\tau) = Z^1(\tau) = \sum c(n) q^n$. The logarithm of the generating function is the generating function for the Hecke transforms of F .

$$\log U(p, q) = \sum_N \frac{1}{N} H_N F(\tau) \quad (19)$$

Notice that the symmetric orbifold of n copies of the Monster CFT still possess a diagonal Monster symmetry. The partition function of the symmetric product can be twisted by this symmetry, and the resulting set of partition functions will clearly be self-consistent (As they are the partition functions of an actual theory). We can collect them into a generating function computed in the spirit of [3] (See also appendix D of [2] for an interesting example). The result is

$$U_g(p, q) = \sum_N p^N Z_g^{\text{Sym}N}(\tau) = \frac{1}{\prod_{n>0} \det_{\mathcal{H}(nm)}(1 - p^n q^n g)} \quad (20)$$

Here $Z_g^1 = \sum q^n \text{Tr}_{\mathcal{H}(n)} g$. The twisted Hecke transforms can be extracted from the logarithm, as

$$(H_N F)_g = \sum_{d|N} \sum_{b=0}^{d-1} F_{g^{N/d}} \left(\frac{N\tau + bd}{d^2} \right) \quad (21)$$

and

$$(H_N F)_{g^a}^{g^c} = \sum_{d|N} \sum_{b=0}^{d-1} F_{g^{cN/d-ab}}^{g^{ad}} \left(\frac{N\tau + bd}{d^2} \right) \quad (22)$$

It is easy to check that $(H_N F)_{g^c}^{g^a}$ have the correct modular transformation rules.

It is quite clear by induction over k that the twisted Hecke operators of the $k = 1$ twisted partition functions $(H_k t)_g$ provide good building blocks for the twisted partition functions

of an hypothetical ECFT with Monster symmetry $Z_{e,g}$.

$$Z_g = \sum a_i(H_i t)_{e,g} \quad (23)$$

This has the appropriate q expansion, with coefficients which decompose into characters $\chi_R(g)$ in a way that is compatible with the decomposition of the coefficients of $Z = \sum a_i H_i J(\tau)$ in dimensions d_R . Moreover the modular transforms

$$Z_{g^a, g^b} = \sum a_i(H_i t)_{g^a, g^b} \quad (24)$$

also have by construction coefficients which decompose into dimensions of irreps of the centralizer of g , with coefficients depending on the numbers a_i , which are pretty much as natural as possible.

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